Improving the Locality of Generalized Integrated Interleaved Code

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Application of Generalized Integrated Interleaved (GII) Codes

- Redundancy is needed to tolerate and recover from failures in distributed storage
- GII codes are a type of locally recoverable erasure codes that substantially reduce the latency and overhead of failure recovery
- Enable short sub-codewords to be adopted for error correction without substantial redundancy increase
- Achieve >40GBytes/sec throughput with excellent correction capability
Outline

- Generalized integrated interleaved (GII) codes
- Modified two-layer GII codes with improved locality
- A generalization of three-layer integrated interleaved codes
- Conclusions
Generalized Integrated Interleaved (GII) Codes

\[
\begin{bmatrix}
\tilde{c}_0 \\
\tilde{c}_1 \\
\vdots \\
\tilde{c}_{v-1}
\end{bmatrix}
= 
\begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & \alpha & \alpha^2 & \cdots & \alpha^{m-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \alpha^{(v-1)} & \alpha^{2(v-1)} & \cdots & \alpha^{(v-1)(m-1)}
\end{bmatrix}
\begin{bmatrix}
c_0 \\
c_1 \\
\vdots \\
c_{m-1}
\end{bmatrix}
\]

Nesting matrix \( G \)

Correction capability

\[
t_v \geq t_{v-1} \geq \cdots \geq t_1 \geq t_0
\]

\[
\mathcal{C}_v \subseteq \mathcal{C}_{v-1} \subseteq \cdots \subseteq \mathcal{C}_1 \subseteq \mathcal{C}_0
\]

\[
\tilde{c}_0 \quad \tilde{c}_1 \quad \tilde{c}_{v-1} \quad \tilde{c}_0, c_1, \cdots, c_{m-1}
\]
Decoding of GII Codes

- Syndromes are needed to correct erasures.
- Higher-order syndromes for the interleaves are generated from the nested syndromes.
- Syndrome conversion matrix is always invertible.
- Each nested syndrome is generated by utilizing all interleaves.

Interleave syndromes
\[
\begin{bmatrix}
S_{j}^{(l_1)} \\
S_{j}^{(l_2)} \\
\vdots \\
S_{j}^{(l_b)}
\end{bmatrix}
= \begin{bmatrix}
1 & 1 & \cdots & 1 \\
\alpha^{l_1} & \alpha^{l_2} & \cdots & \alpha^{l_b} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha^{(b-1)l_1} & \alpha^{(b-1)l_2} & \cdots & \alpha^{(b-1)l_b}
\end{bmatrix}
^{-1}
\begin{bmatrix}
\tilde{S}_{j}^{(0)} \\
\tilde{S}_{j}^{(1)} \\
\vdots \\
\tilde{S}_{j}^{(b-1)}
\end{bmatrix}
\]

$l_1, l_2, \ldots, l_b$: indices of interleaves with more than $t_0$ erasures (exceptional interleaves)
Modified GII Codes

Less powerful nestings involve fewer interleaves

Form the syndrome conversion matrix using the bottom rows of $G'$ as much as possible

- The selected nestings should have sufficient correction capability
- The selected nestings should cover the exceptional interleaves
- Consecutive nestings are used to simplify the selection

The syndrome conversion matrix is invertible if the number of interleaves does not exceed the values in the table

<table>
<thead>
<tr>
<th>$v$</th>
<th>$GF(2^4)$</th>
<th>$GF(2^5)$</th>
<th>$GF(2^6)$</th>
<th>$GF(2^7)$</th>
<th>$GF(2^8)$</th>
<th>$GF(2^9)$</th>
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<tbody>
<tr>
<td>2</td>
<td>16</td>
<td>32</td>
<td>64</td>
<td>128</td>
<td>256</td>
<td>512</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>6</td>
<td>12</td>
<td>22</td>
<td>28</td>
<td>62</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>6</td>
<td>8</td>
<td>12</td>
<td>15</td>
<td>20</td>
</tr>
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</table>
- Modified GII codes preserve the same correction capability as the original GII codes for most practical settings
- Modified GII codes require fewer interleaves to utilize the shared parities when there are fewer extra erasures to correct
- Have very small implementation overhead compared to the GII codes
- Achieve good tradeoff on locality and correction capability
Three-layer Integrated Interleaved Codes

- Layer 2 parities are shared by individual subgroups of interleaves
- Layer 3 parities are shared by all interleaves
- When there are fewer interleaves with fewer extra errors/erasures to correct, layer 2 parities and the other interleaves in the same sub-group are utilized
Nestings in Three-Layer Codes

$G_1$: layer-2 nesting matrix

$G_2$: layer-3 nesting matrix

$\Gamma = \begin{bmatrix} G_1 & 0 & \cdots & 0 \\ 0 & G_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & G_1 \\ G_2 & & & \end{bmatrix}$

Joint nesting matrix

Previous 3-layer nesting

➢ Previous 3-layer integrated interleaved code
  ▪ One-level of nesting in each layer
  ▪ Only one exceptional interleave from each subgroup can be corrected
  ▪ Layer-3 parities only add to the correction of a single exceptional interleave in a subgroup
Generalized Three-layer Integrated Interleaved Codes

\[ \Gamma = \begin{bmatrix} G_1 & 0 & \cdots & 0 \\ 0 & G_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & G_1 \end{bmatrix} \]

\[ G_1 = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \alpha & \alpha^2 & \cdots & \alpha^{m_2-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha^{(v_1-1)} & \alpha^{2(v_1-1)} & \cdots & \alpha^{(v_1-1)(m_2-1)} \end{bmatrix} \]

\[ G_2^i = \begin{bmatrix} 1 & \alpha^{v_1} & \alpha^{2v_1} & \cdots & \alpha^{v_1(m_2-1)} \\ 1 & \alpha^{v_1+1} & \alpha^{2(v_1+1)} & \cdots & \alpha^{(v_1+1)(m_2-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha^{v_1+s_i-1} & \alpha^{2(v_1+s_i-1)} & \cdots & \alpha^{(v_1+s_i-1)(m_2-1)} \end{bmatrix} \]

- \(v_1\): # of levels of codes in layer 2
- \(m_1\): # of subgroups
- \(m_2\): # of interleaves in a subgroup
- \(r\): # of subgroups with extra interleaves to be corrected
- \(s_i\): # of extra interleaves in a subgroup to be corrected

The syndrome conversion matrix formed by the columns corresponding to the interleaves with extra erasures and consecutive rows of \([G_1^T \mid (G_2^i)^T]^T\) in the joint nesting matrix is always invertible, if the numbers of groups and exceptional interleaves in the groups are not exceeded.
Correctable Erasure Pattern Comparisons

Example 1: same redundancy
single-level nesting in 3-layer GII code
Layer-2: \( t_1 = t_0 + 1 \); layer-3: \( t'_1 = t_0 + 2 \)

\[
\begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix}
\]

- Multiple code levels are allowed in layer 2 and 3
- Layer-3 codes can correct additional interleaves
- Layer-3 codes do not have to be stronger than layer-2 codes

Example 2: same redundancy
multi-level nesting in 3-layer GII code
\( \text{II codes: } t_1 = t_0 + 2; \ t'_1 = t_0 + 3 \)
\( \text{GII codes: } t_1 = t_2 = t_0 + 1; \ t'_1 = t_0 + 1; \ t'_2 = t_0 + 2 \)

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<th>GII code correctable by the</th>
<th>GII code but not the</th>
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<td>( {2, 0, 0} ) ( {2, 0, 0} )</td>
<td>( {2, 0, 0} ) ( {2, 0, 0} )</td>
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Implementation Architectures for GII Decoders

- Two-layer GII codes based on Reed-Solomon codes over $GF(2^8)$
  - 8 sub-codewords
  - Length of each sub-codeword: 2040 bit
  - Redundancy: 12.5%

- Hardware complexity
  - <30% area increase compared to hard-decision Reed-Solomon decoder
  - Clock frequency: >550Mhz on FPGA
  - >40GByte/s throughput

Conclusions

- GII codes can achieve hyper throughput with excellent correction capability and low complexity.
- Modified two-layer GII codes improve the locality of erasure correction without degradation on the correction capability for most practical settings.
- Three-layer GII codes further improve the locality of erasure and error correction and the third-layer parities can be used in a flexible way.
- Three-layer GII codes achieve better locality than two-layer GII codes at the cost of higher redundancy.